Temperley-Lieb categories and dynamics in commutative monoids

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Abstract

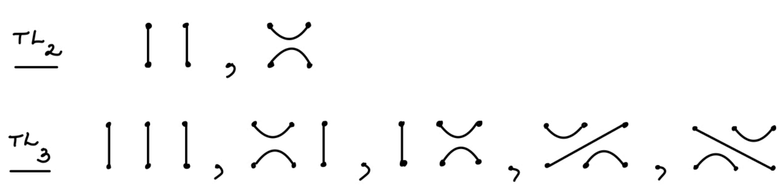
In this work, we explore the Temperley-Lieb algebra and related categories. This paper has three main contributions. (1) First, it introduces the concept of crossingless matchings and provides a little-known proof that for any $n \in \mathbb{N}$, the number of distinct crossingless matchings of $2n$ points is precisely the n^{th} Catalan number. (2) Second, it defines the Temperley-Lieb algebra as it relates to generators and relations and knot invariants, and describes both linear and set-theoretic versions of the Temperley-Lieb category. (3) We then define an extended category $TL_{n,M,\omega} = \mathcal{C}_{M,\omega}$, wherein closed loops are evaluated according to a commutative monoid M and a fixed map of sets $\omega : M \to M$. Our main contribution is an efficient algorithm for planar diagram multiplication which allows us to propose limiting distributions $\mathcal{D}_{M,\omega}$ for the probability distribution $\mathcal{D}^n_{M,\omega}$ on the set of evaluation results $\{t \in M : \bar{x}_1 \cdot x_2 \to_\omega t \text{ for some } x_1, x_2 \in TL_n\}$ as $n \to \infty$. After running simulations for various pairs (M, ω), we conjecture that $\mathcal{D}_{M,\omega} \to Uniform(S)$ for some subgroup $S \subset M$.

Contents

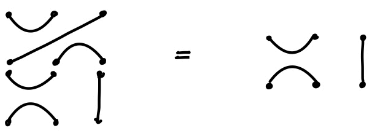
1 Crossingless matchings & Temperley-Lieb diagrams

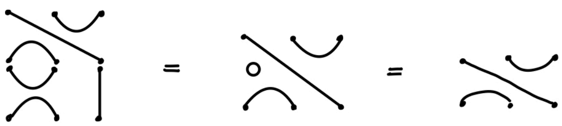
1.1 Temperley-Lieb diagrams

Consider a graph with two rows of n vertices, and assign edges such that each edge connects a dot to exactly one other dot. Moreover, we impose the restriction that the edges, or "arcs," must not cross each other. TL_n, or the monoid of "Temperley-Lieb diagrams" is precisely the collection of all possible crossingless assignments of edges between the $2n$ vertices of this graph.



To familiarize ourselves with TL_n, we begin by noting several properties of crossingless matchings on $2n$ points. First, we define multiplication of any two elements $a, b \in TL_n$ (presented in the two-row representation) by simply placing diagram a vertically above the diagram corresponding to b, and simplifying by following the connected arcs and edges from the top row of the top diagram to the bottom row of the bottom-most diagram to obtain another element of TL_n. For instance, consider the following examples of multiplication in TL_3:

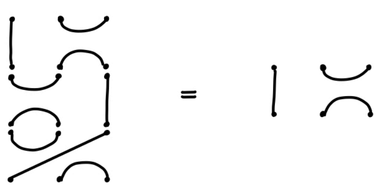




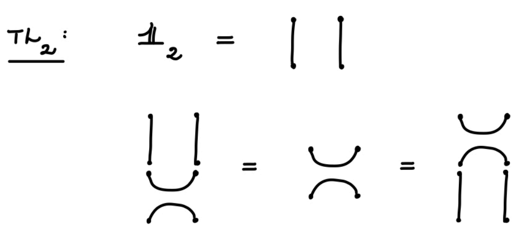
When multiplying two diagrams of TL_n, we occasionally obtain a resultant diagram with loops, as seen in the second example above. In this case, we may take several approaches to resolving these loops. For now, we will simplify the diagram by discarding all closed loops resulting from multiplication; this will ensure that TL_n is closed under multiplication.

Observation 1. *For any fixed $n \in \mathbb{N}$, we have that TL_n is a monoid, i.e. a set equipped with an associative multiplication and an identity element.*

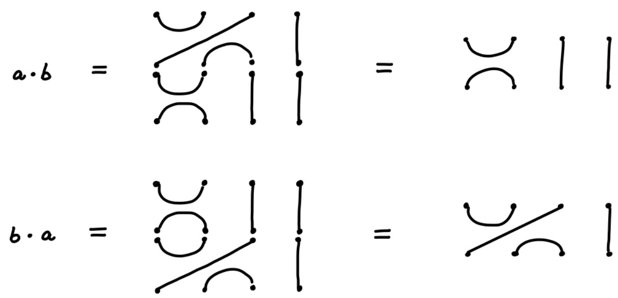
From this definition of multiplication, it is clear that multiplication of TL_n elements is associative: when we stack three diagrams a, b, c to perform multiplication, it does not make a difference whether we first stack a and b together and later adjoin c (and simplify as needed) or if we first stack b and c and adjoin a on top afterwards– this is because we are simply following the edges of the graphs from the top row of the top diagram to the bottom row of the bottom diagram, so different groupings of these diagrams during multiplication will yield the same result.



Moreover, note that $\forall n \in \mathbb{N}$, there is a natural identity diagram in TL_n. We show this explicitly for the small case of TL_2, but for a more general n, the identity $\mathbb{1}_{TL_n}$ is given by the diagram sending any vertex in the top row to the vertex directly beneath it in the bottom row. By our definition of diagram multiplication, it's clear that by stacking this identity above or below any element $\mathcal{D} \in TL_n$ and following the connections from the top row of the top diagram to the bottom row of the bottom diagram, the structure of the resulting simplified diagram must remain as $\mathcal{D}$.



However, multiplication of elements is not necessarily commutative, as seen through the following example:



Additionally, we may ask whether every element $a \in TL_n$ necessarily has an inverse $a^{-1} \in TL_n$ such that $a \cdot a^{-1} = a^{-1} \cdot a = \mathbb{1}_{TL_n}$. – not true, not each element has an inverse.

Remark 2. *We can similarly define multiplication for diagrams given in the one-line representation.*

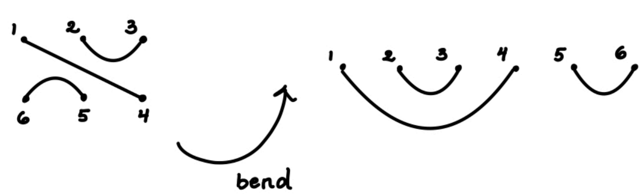
In particular, to multiply two elements $a, b \in TL_n$ provided in the one-line representation, we compute $\bar{a} \cdot b$, where $\bar{a}$ represents the reflection of diagram a about the horizontal axis.

1.2 Counting crossingless matchings with $2n$ endpoints

Naturally, we may first hope to determine the number of possible crossingless matchings for a fixed $n \in \mathbb{N}$. First, it's important to observe that the number of TL_n diagrams is necessarily finite for a fixed n, as we are considering matchings of between a finite number of points.

Embedding a union of intervals into the portion $\mathbb{R} \times [0, 1]$ *of the plane* $\mathbb{R}^2$ *between two horizontal lines. Fix* n *points on the top line* $\mathbb{R} \times \{1\}$ *and* n *points on the bottom line* $\mathbb{R} \times \{0\}$*. The embedding must take the boundary points of the* n *intervals bijectively to these* n *points. We consider such embeddings up to rel boundary isotopies in* $\mathbb{R} \times [0, 1]$.

In order to count the number of crossingless matchings $|TL_n|$ for a fixed n, a common approach is to exploit the recursive structure of TL_n, as shown in [3]. For our (unique) approach, we introduce equivalent diagrammatic representations for crossingless matchings. In particular, we may convert diagrams as follows:

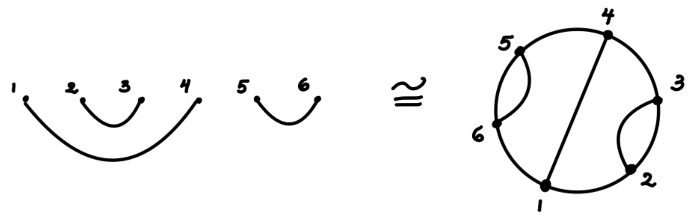


We now provide a proof of the following result, different from the one in [3].

Theorem 3. *For a fixed* $n \in \mathbb{N}$*, the number of crossingless matchings* $|TL_n| = \frac{1}{n+1}\binom{2n}{n} = C_n$*, where* C_n *denotes the* n^{th} *Catalan number.*

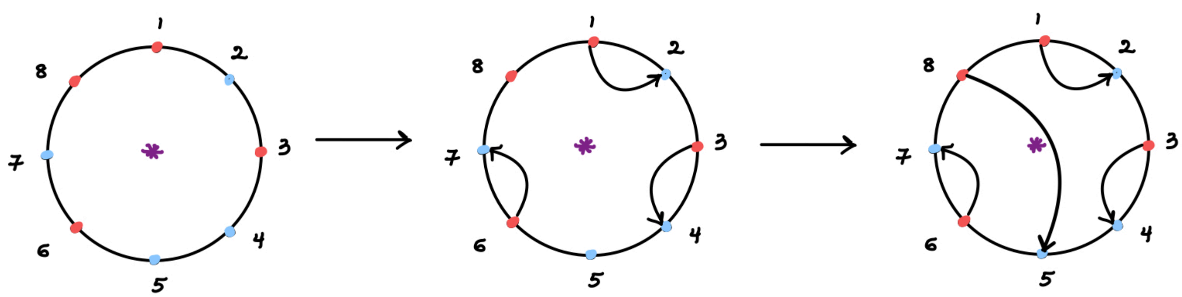
Before we start, observe that there exists a natural bijection between the set of crossingless matchings (or TL_n elements) between two rows of n dots and the set of crossingless matchings of $2n$ dots aligned horizontally. This bijection is given by composing a matching of the first type with a diagram that transforms n points at the bottom of the diagram to n points at the top.

We may also convert from crossingless matchings in the lower half-plane to crossingless matchings in a disk by adding a point at infinity to the half-plane, making it homeomorphic to a disk. This equivalent representation for elements of TL_n will be most useful in our proof of Theorem 3. – tilt disk diagram

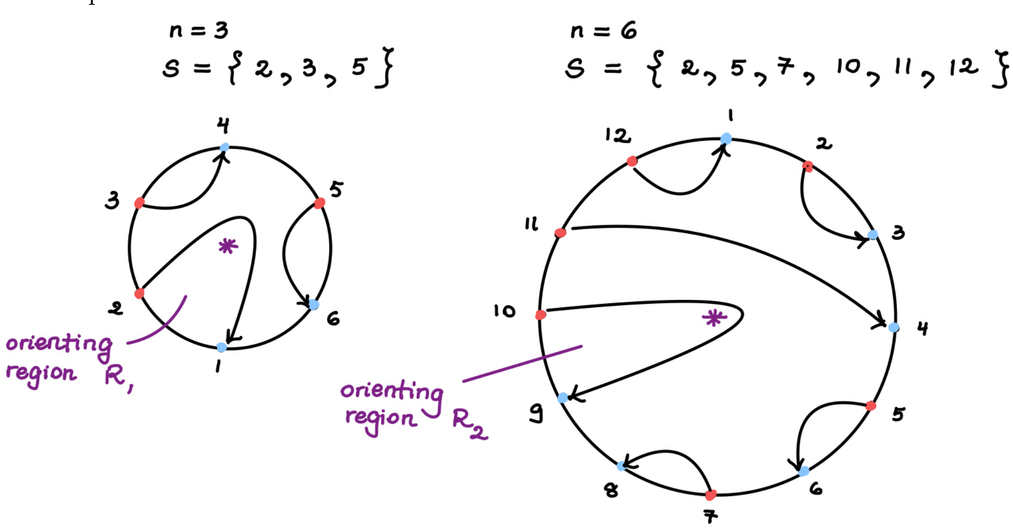


With this in mind, we proceed to provide an alternate proof of the cardinality of $|TL_n|$ for any $n \in \mathbb{N}$.

Proof. First, note that we may generate a crossingless matching embedded into a disk as follows: pick a subset $S \subset \{1, 2, ..., 2n\}$ at random, with $|S| = n$. In the corresponding crossingless matching diagram on the disk, we color vertex i red if $i \in S$ and color vertex i blue otherwise. Now, we may proceed to form the crossingless matching oriented clockwise with respect to the central point (denoted by *) by drawing directed arcs between vertices, always originating from a red vertex and arriving at a blue vertex. To do this, we must first connect any adjacent pairs of blue and red vertices– if we don't do this, it's easy to see that our matching will necessarily feature a crossing, which is forbidden. Once we have connected all adjacent blue and red vertices, we proceed to connect all blue and red vertices which are 2 spaces apart, 4 spaces apart,..., $2n - 2$ spaces apart, until we have successfully added a directed arc from any blue vertex to exactly one red vertex in a crossingless manner. As we mentioned before, we make sure that all arcs are oriented clockwise with respect to the center, as seen in the example below.



For any crossingless matching M with $2n$ endpoints embedded in a disk, observe that $n + 1$ regions of the disk are induced by this matching. We will first argue that the set $\{S : S \subset \{1, 2, .., 2n\}, |S| = n\}$ is in bijection with the set of pairings (M, R), where $M \in TL_n$ and R is the region about which directed arcs between elements are oriented in a clockwise orientation. To better understand what this means, let's look at an example:



To show this bijection, we proceed as follows: pick any subset $S \subset \{1, 2, ..., 2n\}$, color points $i \in S$ in red, and color all remaining points $i \notin S$ in blue. Then, add directed arcs by applying the algorithm described above, where arcs are oriented clockwise with respect to a point in the center (as above). We notice that by this construction, we obtain a region R which contains the fixed point p such that arcs are oriented clockwise with respect to this R.

Alternatively, choose any crossingless matching $M \in TL_n$, and pick a region R of the disk induced by this matching. Then, we may construct a directed crossingless matching (M, R) by orienting the preexisting arcs to be clockwise with respect to R.

At this point, we have shown that there exists a bijection between the set $\{S : S \subset \{1, 2, .., 2n\}, |S| = n\}$ is in bijection with the set of pairings pairings (M, R) with $M \in TL_n$ and R any region of the matching induced by M in the disk. In particular, it follows that

$$\#(\text{crossingless matchings in } TL_n) \cdot (n+1) = \binom{2n}{n}$$

So, we have shown that

$$|TL_n| = \frac{1}{n+1}\binom{2n}{n}$$

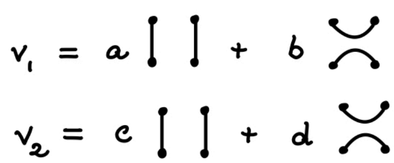
as desired. □

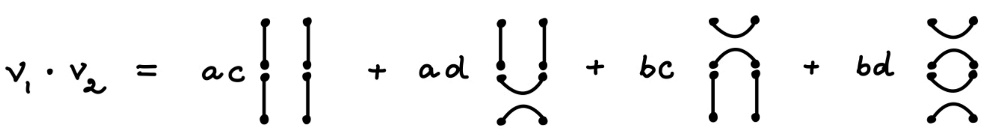
2 Temperley-Lieb algebra and Temperley-Lieb monoids

At this point, having examined the properties and structure of TL_n, we are ready to define the Temperley-Lieb Algebra $TL_n(x)$ for $n \in \mathbb{N}$ and $x \in \mathbb{C}$. Specifically, we view $TL_n(x)$ as the vector space whose basis elements are diagrams in TL_n, i.e.

$$TL_n(x) = \text{span}\{\mathcal{D} \in TL_n\} = \{\sum_{\alpha_\mathcal{D}} \alpha_\mathcal{D} \mathcal{D} | \alpha_\mathcal{D} \in \mathbb{C}\}$$

For instance, the algebra $TL_2(x)$ represents a vector space with basis elements in TL_2:

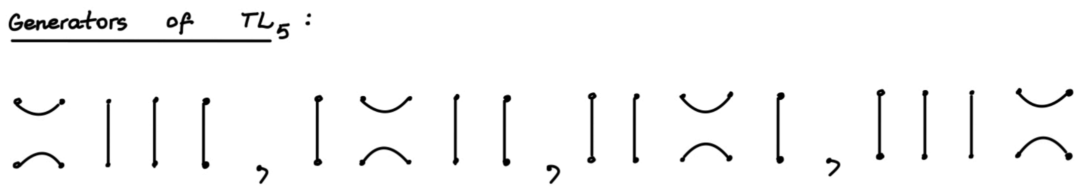




At this point, it's important to address a key difference between simplifications in TL_n and in $TL_n(x)$: recall that in TL_n multiplication, we dropped any loops which resulted from the product of diagrams. In $TL_n(x)$, we will count the number of dropped loops before removing them from the resulting vector– specifically, for every component of the vector product, we multiply the component by x^l, where l denotes the number of dropped loops when multiplying TL_n diagrams for this term. In particular, simplifying the above example, we see that

2.1 The Temperley-Lieb Algebra via Generators and Relations

Next, we describe an alternate representation for the Temperley-Lieb algebra. For ring R and $x \in R$, we define $TL_n(x)$ as the associative R-linear algebra of dimension $n \in \mathbb{N}$ with $n-1$ generators denoted as $U_1, ..., U_{n-1}$, where U_i is the diagram with an arcs connecting points i and $i+1$ in the top and bottom rows, and all other arcs connecting points in the top row to points directly under them in the bottom row.



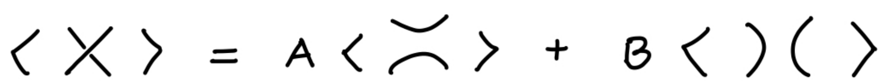
In fact, we can check that the following properties of generators are satisfied:

- $U_iU_i = xU_i$
- $U_iU_{i+1}U_i = U_i$ for $|i-j| = 1$
- $U_iU_j = U_jU_i$ for $|i-j| > 1$
- $xU_i = U_ix$ for $x \in R$

Now, using the above multiplicative relations, we may define $TL_n(x)$ via R-linear combinations of generators and elements of R, i.e. for any $D \in TL_n(x)$, we may write $D = \sum_i r_iU_i$ for $r_i \in R$.

2.2 The Temperley-Lieb algebra and knot invariants

At this point, one may wonder how the Temperley-Lieb algebra relates to knots. In fact, we can consider elements of $TL_n(x)$ as "tangle" diagrams with the following relation, known as the *bracket polynomial*:



where $B = A^{-1}$. Additionally, to represent the Temperley-Lieb algebra using tangles, we add the relation that for any tangle K, we have $K \cup \circ = xK$, where $x = -A^2 - A^{-2}$. Through these relations, we make two observations: first, for any tangle with a crossing, we may replace it with a linear combination of two tangles with crossings resolved. Second, we see that any unconnected loop may be removed from the diagram if we multiply the diagram by $x = -A^2 - A^{-2}$, see [1].

Notably, these relations allow us to replicate the structure of the Temperley-Lieb algebra within the context of knot invariants. In fact, for any generator σ_i of the braid group $\mathcal{B}_n$, we obtain an element of $TL_n(x)$ via the homomorphism defined as

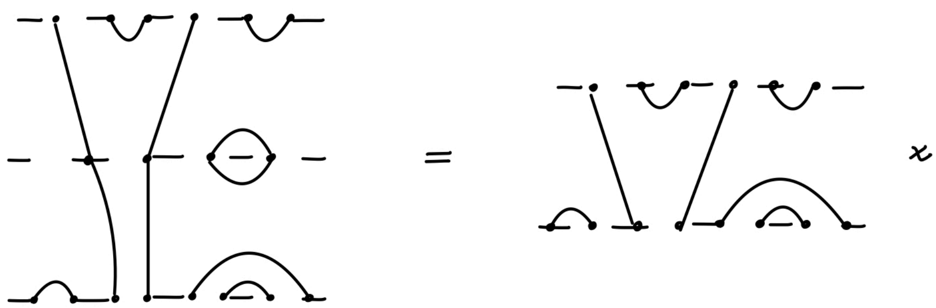
$$\rho_n : \mathcal{B}_n \to TL_n(x)$$
$$\rho_n(\sigma_i) = A(1_n) + A^{-1}U_i$$
$$\rho_n(\sigma_i^{-1}) = A^{-1}(1_n) + A(U_i)$$

Closing a braid results in a link, and the algebraic analogue of this closure is the Jones trace on the Temperley-Lieb algebra. The Jones trace is given by connecting the top n points of a Temperley-Lieb diagram to the bottom n points by n disjoint arcs in the plane and then evaluating the resulting diagram via Kauffman's rules. The trace of $\rho_n(\sigma)$, for a braid σ, equals the Jones polynomial (the Kauffman bracket) of the link $\widehat{\sigma}$, which is the closure of σ.

2.3 The Temperley-Lieb category

Define a $\mathbb{C}$-linear category $\mathcal{C}$ that extends the Temperley-Lieb algebra as follows: let objects of $\mathcal{C}$ be natural numbers $n \in \mathbb{N}$ and let morphisms from m to n be $\mathbb{C}$-linear combinations of diagrams which connect the top row of n vertices and the bottom row of m vertices by arcs pairwise, in a similar way as in our earlier discussion of TL_n. Isotopic rel boundary diagrams describe equal morphisms, and any circles in diagrams evaluate to x.

To compose any two morphisms $f : \mathbf{n} \to \mathbf{m}$ and $g : \mathbf{m} \to \mathbf{l}$, we identify the bottom row of m vertices of f with the top row of m vertices of g, and simplify the paths as we did earlier when we multiplied crossingless matchings in TL_n. Observe that through this construction, we recover elements of TL_n as the endomorphism monoid $\mathrm{End}(n, n)$ for $n \in \mathbb{N}$.



There is also a set-theoretic version $\mathcal{C}_S$ of the Temperley-Lieb category. It has the same objects $n \in \mathbb{N}$ as the Temperley-Lieb category, and the morphisms from n to m are planar diagrams of arcs and circles as before, up to rel boundary isotopies. Now, however, circles are not evaluated and a diagram may be an arbitrarily complicated configuration of nested circles in each of the regions separated in the plane strip by its arcs. In this category no linear combinations of diagrams are formed – morphisms are individual diagrams.

3 Evaluating Planar Diagrams & Associated Categories

3.1 Evaluating planar diagrams

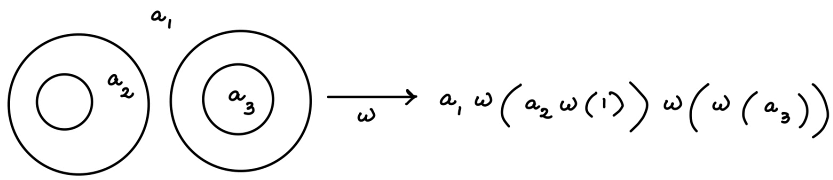
Let M be a commutative monoid and $\omega : M \to M$ any map on M viewed as a set. One can study planar diagrams as encodings of elements of monoid M. Suppose D is a planar diagram consisting of finitely many (possibly nested) circles and elements of M floating in the plane. To such a diagram, we associate an element $\langle D \rangle$ of M inductively on the complexity of the diagram, as follows:

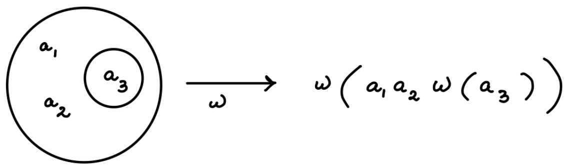
1. If D' is given by a single empty circle, then $\langle D' \rangle = \omega(1)$, where $1 \in M$ is the unit element.

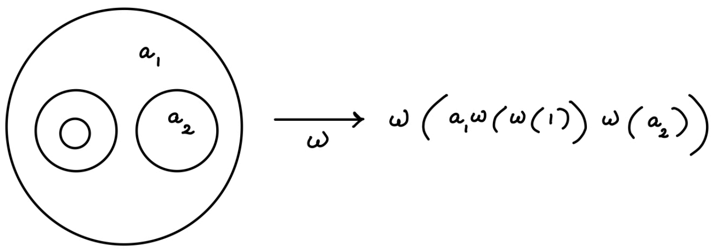
2. If D' is given by a circle wrapped around diagram D and $a \in M$ is an element floating outside of D but inside of D' then $\langle D' \rangle = \omega(a)\langle D \rangle$.

3. If D' is given by a circle wrapped around diagram D then $\langle D' \rangle = \omega(\langle D \rangle)$.

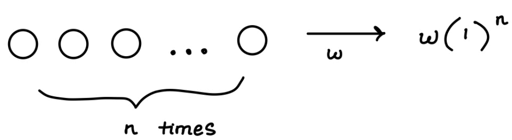
4. If D' is a union of diagrams D_1, D_2 floating in disjoint disks in $\mathbb{R}^2$ then $\langle D' \rangle = \langle D_1 \rangle \langle D_2 \rangle$.

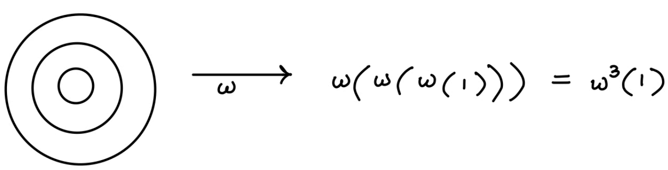


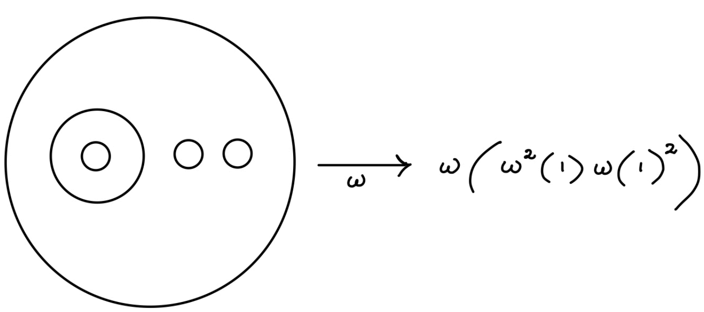




Note that we require that M is commutative, since we do not have an ordering on diagrams D_1, D_2 in case (4) when multiplying their evaluations. As a special case, given a collection C of circles embedded in the plane, we can evaluate it to $\langle C \rangle \in M$. In particular, as seen in the diagram below, for a planar diagram C_1 consisting of n parallel disjoint empty circles, ω associates the value $\langle C_1 \rangle = \omega(1) \cdot \omega(1) ... \cdot \omega(1) = \omega^n(1)$ to the diagram C_1. Similarly, in the second example below, diagram C_2 consists of three nested circle and ω associates the value $\omega(\omega(\omega(1)))$ to $\langle C_2 \rangle$, thereby reflecting the circle nesting structure of the diagram through value of the evaluation in M. The last example $\langle C_3 \rangle$ is evaluated in a similar manner.







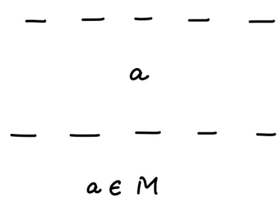
3.2 Set-theoretic Temperley-Lieb category for (M, ω)

As before, let M be a commutative monoid, and $\omega : M \to M$ be a map on M. We define a category $\mathcal{C}_{M,\omega}$ (Temperley-Lieb category extended via (M, ω)), where objects are natural numbers $\mathbb{N} = \{0, 1, 2, \dots\}$ [2]. The set of morphisms $\mathsf{Hom}_{\mathcal{C}}(n, m)$ is nonempty if and only if $n + m$ is even, and in this case, it consists of diagrams in $\mathbb{R}^2 \times [0, 1]$ made of $\frac{n+m}{2}$ disjoint arcs connecting n fixed points on $\mathbb{R}^2 \times \{0\}$ to m fixed points on $\mathbb{R}^2 \times \{1\}$. These diagrams are viewed up to isotopy. Furthermore, an element of M floats in each of $\frac{n+m}{2} + 1$ regions of the diagram (element 1 can be omitted). If M is finite, $\mathsf{Hom}(n, m)$ has cardinality $c_k |M|^{k+1}$, where $k = C_{\frac{n+m}{2}}$ is the corresponding Catalan number.

Let us now give examples of morphisms in this category and then explain how to compose them (composition is given by concatenating diagrams and inductively simplifying the resulting diagram).

Example 4. *Consider* $\mathsf{Hom}_C(0, 0)$.

Since we consider diagrams connecting 0 points on the top row to 0 points in the bottom row, there is a single region in which elements of M may float. So, any morphism $f \in \mathsf{Hom}_C(0, 0)$ may be represented as follows:



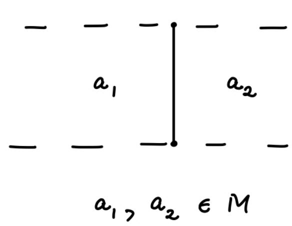
Notice that if $n > 1$ elements $a_1, ..., a_n$ were to float in the single region above, it follows by the construction in 3.1 that the diagram is equivalent to one with just the element $a_1 \cdot a_2 \cdot ... \cdot a_n$ floating in the region, as in the figure above. Since M is closed under multiplication it follows that $\mathsf{Hom}_C(0, 0) \cong M$.

Example 5. *Examine* $\mathsf{Hom}_C(0, 1)$

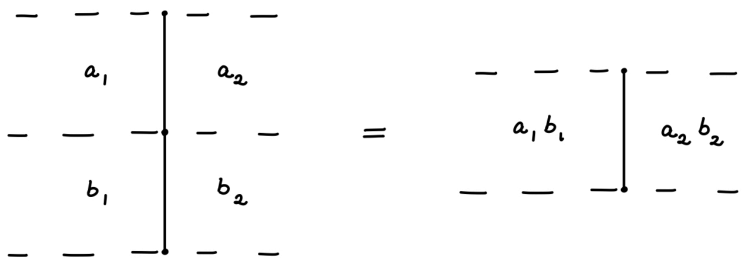
Notice that for this case, we consider all possible crossingless matchings of 0 points in the top row with 1 point in the bottom row. This does not allow for any valid matchings, so $\mathsf{Hom}_C(0, 1) = \emptyset$.

Example 6. *Consider* $\mathsf{Hom}_C(1, 1)$.

Here, consider diagrams connecting 1 point in the top row with 1 point in the bottom row, allowing for morphisms of the following form:



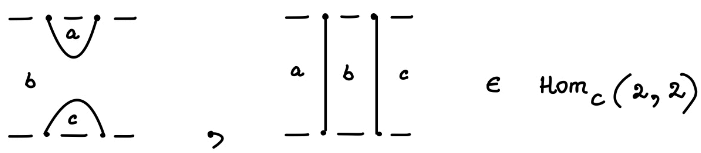
Observe that there is a trivial isomorphism $\mathsf{Hom}_C(1, 1) \cong M \times M$, where the diagram with a_1 in one region and a_2 in another region corresponds to the element $(a_1, a_2) \in M \times M$. Moreover, note that $\mathsf{Hom}_C(1, 1)$ is closed under composition, where composition of two morphisms f and g is obtained by stacking the diagram for f on top of the diagram for g and following corresponding arcs to simplify the result.



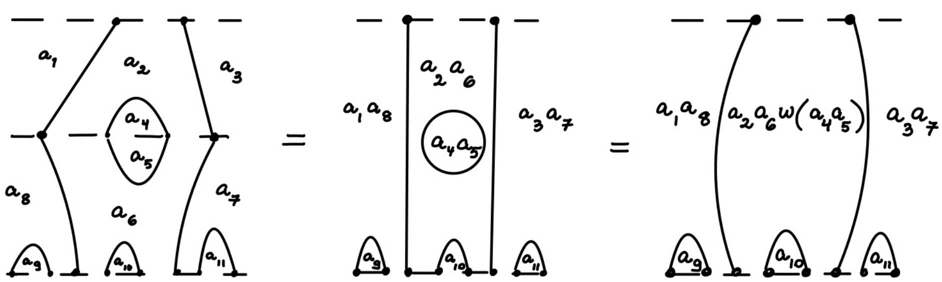
Additionally, by the brief computation above, $\mathsf{Hom}_C(1,1)$ is commutative for any commutative M. This is not true in general, as we will see in the next example.

Example 7. *Consider* $\mathsf{Hom}_C(2,2)$.

We see that in this case, $\mathsf{Hom}_C(2,2)$ has morphisms of two distinct forms:

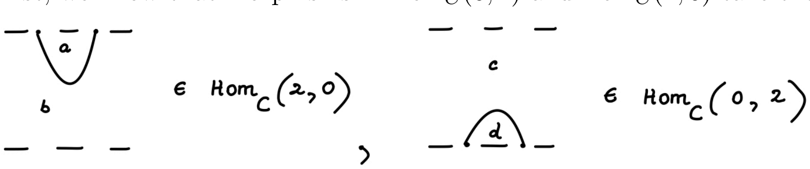


Generally, for any morphisms $\alpha \in \mathsf{Hom}(n_1, n_2)$ and $\beta \in \mathsf{Hom}(n_2, n_3)$, the composition $\beta\alpha \in \mathsf{Hom}(n_1, n_3)$ is defined as follows. Concatenate α and β to get a planar diagram of arcs, circles and elements of M. Inductively simplify the diagram as explained in Section 3.1. This simplification allows, in particular, to inductively remove all circles from the concatenated diagram and reduce objects floating in a region of the concatenated diagram bounded by arcs to a single element of M. In the following example, we compose an element of $\mathsf{Hom}_C(2,4)$ with an element of $\mathsf{Hom}_C(4,8)$ and obtain an element of $\mathsf{Hom}_C(2,8)$.

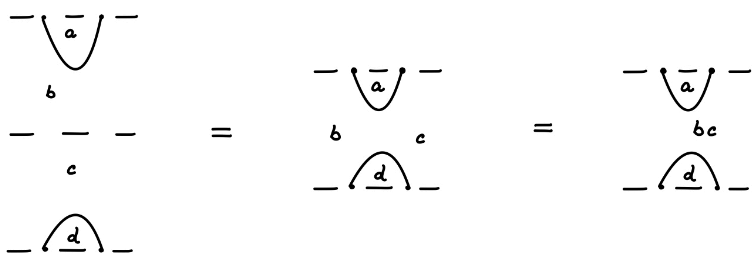


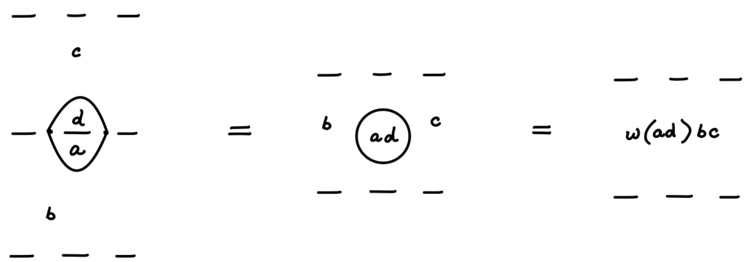
Example 8. *Next, we explore* $\mathsf{Hom}_C(0,2)$ *and* $\mathsf{Hom}_C(2,0)$.

First, we know that morphisms in $\mathsf{Hom}_C(0,2)$ and $\mathsf{Hom}_C(2,0)$ take the following forms:



In this case, we see that $\mathsf{Hom}(2,0) \times \mathsf{Hom}(0,2) \to \mathsf{Hom}(2,2)$, while $\mathsf{Hom}(0,2) \times \mathsf{Hom}(2,0) \to \mathsf{Hom}(0,0)$, as seen diagramatically in the next few figures.



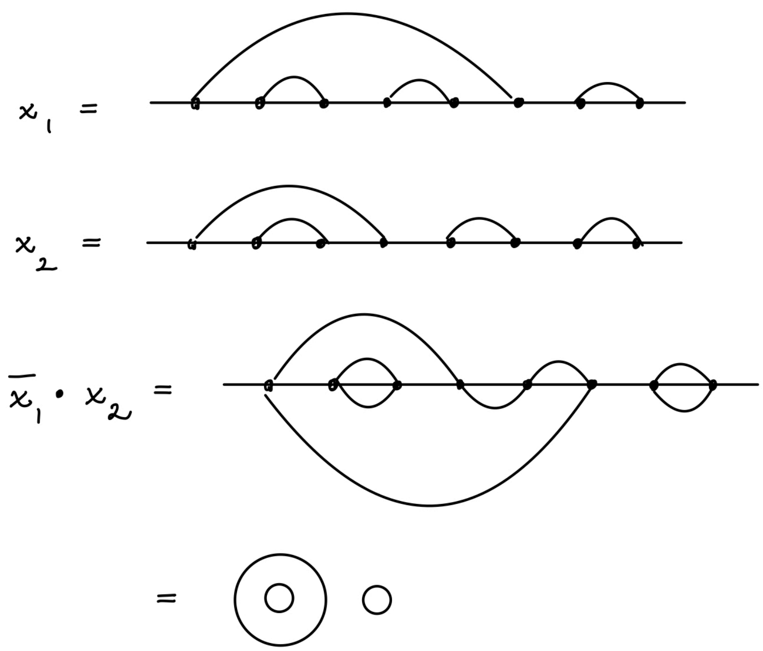


To see the relationship with the Temperley-Lieb category described in section 2.3, we define a functor $F : \mathcal{C}_S \to \mathcal{C}_{M,\omega}$ as follows: map any morphism $f : \mathbf{n} \to \mathbf{m}$ of $\mathcal{C}_S$ to a morphism $f_{M,\omega} : \mathbf{n} \to \mathbf{m}$ of $\mathcal{C}_{M,\omega}$, where $f_{M,\omega}$ is the same as morphism f, except that any closed loops appearing in f are evaluated recursively via (M, ω).

4 Studying limiting distributions via computational methods

In this section, we discuss the experimental portion of our study and the accompanying results. From an earlier discussion addressing multiplication in TL_n, multiplying $x_1, x_2 \in TL_n$ for a particular $n \in \mathbb{N}$ results in some loops. These loops need to be dropped in order to ensure closure under multiplication for TL_n.

However, let us instead keep all loops in the resulting diagrams, and presume that multiplication takes place with x_1 and x_2 represented as crossingless matchings of $2n$ points aligned horizontally. For example, consider the example in the figure below:



Now, for a fixed pair (M, ω) for any monoid M and map $\omega : M \to M$, we recursively evaluate innermost closed loops found in the diagram via $\circ \to_\omega \omega(1)$. A loop that circles a product $m \in M$ of elements of M is evaluated to $\omega(m)$. Denote this evaluation of the closed diagram $\bar{x}_1 \cdot x_2$ by $ev(x_1, x_2)$. In the above example, $ev(x_1, x_2) = \omega(\omega(1))\omega(1)$. This gives us a map

$$TL_n \times TL_n \longrightarrow M, \quad (x_1, x_2) \longmapsto ev(x_1, x_2) \in M.$$

Assuming that $x_1, x_2 \in TL_n$ are distributed uniformly, we get a probability distribution on M for each $n \geq 1$. Let $\mathcal{D}^n_{M,\omega}$ denote the probability distribution for the set of evaluation results $\{t \in M : \bar{x}_1 \cdot x_2 \to_\omega t$ for some $x_1, x_2 \in TL_n\}$. Now, we are ready to pose our main question:

Question 9. *For a fixed finite commutative monoid M and a map $\omega : M \to M$, is there a limiting probability distribution $\mathcal{D}_{M,\omega}$ such that $\mathcal{D}^n_{M,\omega} \to \mathcal{D}_{M,\omega}$? In other words, for any $t \in M$, what does $\mathbb{P}_n(t)$ converge to as $n \to \infty$ in this evaluation model?*

Motivated by this question, we proceed to conjecture limiting distributions $\mathcal{D}_{M,\omega}$ for various pairs (M, ω) based on evidence from computer simulations. We will describe our approach in greater detail in the following sections.

4.1 Conjectured limiting distributions

In attempt to learn more about the limiting distribution $\mathbb{P}(t)$ under this evaluation model, we ran simulations for numerous pairs (M, ω). These results (as well as the precise probabilities) are listed in the appendix. In the next few paragraphs, we will discuss our observations and provide heuristic explanations for the conjectured limiting distributions for specific pairs (M, ω). We begin by studying commutative monoids of size 2. Note that if $\omega(1) = 1$, then all closed diagrams would be evaluated to 1. To avoid this trivial case, we pick maps ω under the restriction that $\omega(1) \neq 1$.

Conjecture 10. *Let $M_1 = \{1, a : a^2 = 1\}$ and $M_2 = \{1, a : a^2 = a\}$ be the two monoids of order two (one for each isomorphism class). Consider the map ω given by $\omega(1) = a$ and $\omega(a) = 1$, for each of the two monoids. We conjecture that $\mathcal{D}^n_{M_1,\omega} \to Uniform(M_1)$ and $\mathcal{D}^n_{M_2,\omega} \to \begin{cases} a & w.p.\ 1 \\ 1 & w.p.\ \ 0 \end{cases}$.*

We observed the convergences stated above through repeated computer simulations. Intuitively, these experimental results can be explained as follows. Monoid M_1 is a group, which encourages good mixing of elements in M_1, so we can expect both 1 and a to appear equally often (as $a \cdot a = a^2$ maps back to 1 and $1 \cdot a = a$). Similarly, for monoid M_2, as $n \to \infty$, suppose we are given a diagram $\overline{x}_1 \cdot x_2$. It has some number (one or more) of outer circles. Each of them evaluates to either 1 or a, and $\overline{x}_1 \cdot x_2$ evaluates to the product of these evaluations. Even a single outer circle evaluating to a will result in the product equaling a, since $a^2 = a$. Experiments show that for large n probability of having few outer circles go to 0. So, it follows that a diagram having many other circles evaluates to a with high probability. Next, we will consider commutative monoids of size 3.

Conjecture 11. *Let $M = \{1, a, a^2 : a^3 = a^2\}$ and pick any map ω such that $\omega(1) \neq 1$. We conjecture that $\mathbb{P}(a^2) \to 1$, $\mathbb{P}(a) \to 0$ and $\mathbb{P}(1) \to 0$.*

When we ran our code this example for large n, we observed that $\mathbb{P}(a^2) \to 1$, while $\mathbb{P}(a) \to 0$ and $\mathbb{P}(1) \to 0$. Heuristically, we can explain this behavior by the $a^3 = a^2$ condition of the monoid, which causes any "overflow" a^3 to map back to a^2, thereby obtaining element a^2 more often than others. Alternatively, we can see this as follows: it is clear that as the number of outer circles gets larger, with high probability we will get that at least one circle evaluates to a^2 or that at least 2 outer circles evaluate to a, which then results in an a^2 evaluation overall ($a \cdot a = a^2$, and $a^2 \cdot 1 = a^2 \cdot a = a^2 \cdot a^2$).

Conjecture 12. *For $M = \{1, a, a^2 : a^3 = 1\}$ and any map ω with $\omega(1) \neq 1$, $\mathcal{D}^n_{M,\omega} \to Uniform(M)$.*

Likewise, we may adjust the above monoid M by replacing the condition that $a^3 = a^2$ by $a^3 = 1$, thereby allowing for better mixing within the monoid $M = \{1, a, a^2 : a^3 = 1\}$. In this case, we observed that for any map ω, $\mathcal{D}^n_{M,\omega} \to Uniform(M)$ as $n \to \infty$. Heuristically, this may be explained by the good mixing properties and group structure of the monoid, allowing all elements to be reached "equally often."

Conjecture 13. *For $M = \{1, a, a^2, a^3 : a^4 = a\}$ and any map ω with $\omega(1) \neq 1$, $\mathcal{D}^n_{M,\omega} \to Uniform(S)$ where $S = \{a, a^2, a^3\}$ is a subgroup of M.*

As we examine the structure of monoid $M = \{1, a, a^2, a^3 : a^4 = a\}$, we see that as $n \to \infty$, at least one outer circle evaluates to a with high probability, so the entire evaluation will belong to S (as $a \cdot b \in S$ for any $b = a^l \in M$). Intuitively, we may attribute the uniformity of the observed limiting distribution to the good mixing properties of M on subgroup S. In fact, we generalize Conjecture 13 as follows:

Conjecture 14. *Fix any $m \in \mathbb{N}$. For $M = \{1, a, a^2, a^3, ..., a^m : a^{m+1} = a^k$ for some $0 \leq k \leq m\}$ and any map ω with $\omega(1) \neq 1$, $\mathcal{D}^n_{M,\omega} \to Uniform(S)$ where $S = \{a^k, ..., a^m\}$ is a subgroup of M.*

For similar reasons as above, we see that the property $a^{m+1} = a^k$ for some $0 \leq k \leq m$ ensures that if at least one outer circle evaluates to a^k, then the entire evaluation will be inside of subgroup $\{a^k, ..., a^m\}$. With high probability, at least one outer circle will evaluate to a^k as $n \to \infty$.

Conjecture 15. *For $M = \{1, a, b : ab = ba = b, a^2 = a, b^2 = b\}$ and any map ω with $\omega(1) \neq 1$, $\mathcal{D}^n_{M,\omega} \to$*
$$\begin{cases} b & w.p.\ 1 \\ a & w.p.\ 0 \\ 1 & w.p.\ 0 \end{cases}.$$

Here, we see that as $n \to \infty$, at least one outer circle will evaluate to b with high probability. Furthermore, note that $b \cdot d = b$ for any element $d \in M$. So, the evaluation gets "trapped" on b with high probability, thus explaining the observed convergence.

Conjecture 16. *For $M = \{1, a, a^2, b : a^3 = a, b^2 = b, ab = a\}$ and any map ω with $\omega(1) \neq 1$, $\mathcal{D}^n_{M,\omega} \to Uniform(S)$ where $S = \{a, a^2\}$ is a subgroup of M.*

Notice that in this case, any element c multiplied by a yields a result in $\{a, a^2\}$, as $a \cdot b = a$, $a \cdot a = a^2$, and $a \cdot a^2 = a^3 = a$. Moreover, we see that at least one outer circle will evaluate to a with high probability as $n \to \infty$, so the overall evaluation will be in $\{a, a^2\}$ with high probability as well.

4.2 Algorithm for Planar Diagram Multiplication

It's important to note that the results and conjectures stated above were generated using a computer simulation in Python (linked here: `https://github.com/eg314159/TL_distribution`). In this section, we highlight some of the key ideas going into our planar diagram multiplication algorithm.

On first glance, one might assume that our simulation must compute $\bar{x}_1 \cdot x_2$ for every pair $x_1, x_2 \in TL_n$ for a fixed $n \in \mathbb{N}$. However, it is not computationally feasible to simply compute all possible products of pairs of elements of TL_n for $n \in \mathbb{N}$, as we have shown that $|TL_n| = \frac{1}{n+1}\binom{2n}{n} = \Theta(\frac{2^{2n}}{\sqrt{n}(n+1)})$ by Sterling's approximation, and is thus this would be very computationally intensive. In attempt to reduce running time and capture the behavior of the limiting distribution, we chose to implement a simulation wherein for various $n \leq B$ for some large enough $B \in \mathbb{N}$, a subset of N distinct elements $\mathcal{S}$ is selected uniformly at random from TL_n, and for every $x_1, x_2 \in \mathcal{S}$, we compute the diagrammatic product $\bar{x}_1 \cdot x_2$ and record the corresponding evaluation under a chosen pair (M, ω).

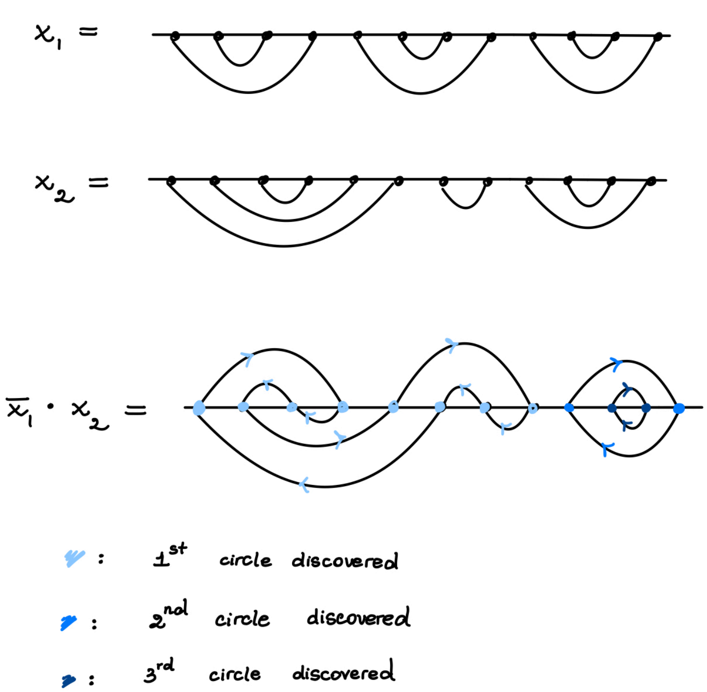
Now, how do we implement a diagrammatic product $\bar{x}_1 \cdot x_2$? We proceed as follows:

1. Label $2n$ endpoints by integers $1, 2, ..., 2n$.

2. Store the arcs (i, j) of x_1 in a bidirectional dictionary *upper-arcs*, (by recording a key-value mapping between i and j), and store the arcs of x_2 in another bidirectional dictionary *lower-arcs* (recording the arc as before). Note that this naming is suggestive of the way in which we multiply diagrams with $2n$ endpoints on the real line, as we flip the arcs of x_1 and stack $\bar{x}_1$ on top of x_2, following arcs as needed to simplify into a diagram of nested circles.

3. Define an array *unexplored* $= \{1, 2, ..., 2n\}$ of vertices that have not yet been visited by the algorithm.

4. Now, beginning from the smallest unexplored vertex v_s, we use dictionaries *upper-arcs* and *lower-arcs* (in alternating order, as an upper arc is always followed by a lower arc and vice versa) to follow the arcs from v_s until v_s is encountered again: $v_s \to v_{i_1} \to v_{i_2} \to ... \to v_{i_k} \to v_s$. Note that such a loop will always be found if x_1 and x_2 are well-defined crossingless-matchings in TL_n. Once v_s is encountered, we record the vertices which were visited, remove them from our array *unvisited*, and record that a circle was found passing through vertices $v_{i_1}, ..., v_{i_k}, v_s$.



5. Repeat step 3, beginning with the next smallest unvisited vertex and traversing the arcs as before. Each time a circle is found, record the nesting with respect to other circles which have already been detected—this can be done easily by comparing the smallest and largest vertices of the current circle with the smallest and largest vertices of the previous circle which was detected.
6. Finally, once all vertices have been visited, we recursively reconstruct the nesting of detected circles. This yields the resulting closed diagram corresponding to the product $\bar{x}_1 \cdot x_2$, as desired.

By applying the above procedure, we compute all closed diagrams corresponding to pairwise products from our random sample $\mathcal{S}$ of TL_n. Having obtained these product diagrams, we recursively evaluate the closed diagrams with respect to (M, ω).

Observation 17. *For any fixed $x_1, x_2 \in TL_n$, the above algorithm computes the diagrammatic product (and simplified circle-nesting configuration) in $O(n)$ time.*

In particular, the multiplication procedure only examines each vertex once, and for each vertex it performs one look-up operation in the appropriate dictionary *upper-arcs* or *lower-arcs*. Thus, it's clear that the algorithm for multiplying $\bar{x}_1 \cdot x_2$ (for just one fixed pair) runs in linear-time.

4.3 Open problems and questions

In the future, we are interested in investigating different (M, ω) pairs as well as the associated limiting distributions $\mathcal{D}_{M,\omega}$ (if they exist). While this work proposes conjectured limiting distributions with strong evidence from computer simulations, we do not provide rigorous proofs of the observed convergences – this would also be an important avenue for future research in this area. Moreover, as proposed in Conjectures 14 and 15, we have found strong evidence that for any (M, ω) pair, $\mathcal{D}^n_{M,\omega} \to Uniform(S)$, where $S \subset M$ is a subgroup determined by properties of M and ω. The inclusion $S \subset M$ is, in general, not unital, taking the unit element of S to an idempotent in M. Motivated by this, we pose the question (for future study):

Question 18. *Are there examples (M, ω) such that $\mathcal{D}_{M,\omega} \neq Uniform(S)$ for a suitable subgroup $S \subset M$?*

Based on our experiments, it seems that the answer is no, but we are interested in exploring this question more formally in our future work. Lastly, we pivot from considering closed diagrams evaluated via (M, ω) pairs. Fix $n \in \mathbb{N}$, and consider all possible closed diagrams produced by multiplying $\bar{x}_1 \cdot x_2$ for $x_1, x_2 \in TL_n$. Let $\mathcal{B}_n$ be the probability distribution for the number of outer circles over all elements generated through pairwise multiplication $\bar{x}_1 \cdot x_2$. Normalize $\mathcal{B}_n$ to the function $\overline{\mathcal{B}}_n(x) = \sqrt{n}\mathcal{B}_n(\sqrt{n}x)$. This function is nonzero only at points $\frac{1}{\sqrt{n}}, \frac{2}{\sqrt{n}}, \ldots, \frac{n}{\sqrt{n}}$ and the sum of its values at these points is $\sqrt{n}$.

Question 19. *Does the sequence of functions $\overline{\mathcal{B}}_n$ converge to some continuous limiting distribution $\mathcal{B}$ as $n \to \infty$?*

5 Summary

Through this paper, we provide an introduction to crossingless matchings and the Temperley-Lieb algebra. We begin by defining operations on crossingless matchings, and we provide a little-known proof that the number of crossingless matchings on $2n$ points is precisely the n^{th} Catalan number $\frac{1}{n+1}\binom{2n}{n}$. Next, we define both the linear and set-theoretic Temperley-Lieb categories $\mathcal{C}$ and $\mathcal{C}_S$, and we build a functor $F : \mathcal{C}_S \to \mathcal{C}_{M,\omega}$ wherein closed loops are evaluated recursively via (M, ω). After describing properties of both categories, for fixed pairs (M, ω), we move to consider the limiting distribution $\mathcal{D}_{M,\omega}$ of distributions $\mathcal{D}^n_{M,\omega}$ on evaluations $\{t \in M : \bar{x}_1 \cdot x_2 = t \text{ for some } x_1, x_2 \in TL_n\}$. To investigate these limiting distributions, we devise a linear-time algorithm for TL_n multiplication by exploiting properties of crossingless matchings. Due to abundant evidence from computer simulations, we conjecture that for any $m \in \mathbb{N}$, a monoid $M = \{1, a, ..., a^m : a^{m+1} = a^k \text{ for some } 0 \leq k \leq m\}$, and map ω such that $\omega(1) \neq 1$, we have that $\mathcal{D}^n_{M,\omega} \to \mathcal{D}_{M,\omega} = Uniform(S)$ where $S = \{a^k, ..., a^m\}$ is the largest subgroup of M. In the future, we hope to study a more general version of this conjecture; in particular, for any finite commutative monoid M and any map of sets $\omega : M \to M$ such that $\omega(1) \neq 1$, does $\mathcal{D}^n_{M,\omega} \to \mathcal{D}_{M,\omega} = Uniform(S)$ for some subgroup $S \subset M$?

6 Acknowledgement

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7 Appendix

In the next few pages, we provide results generated by computer simulations for various pairs (M, ω). Note that in the left-most column, we provide the monoids which we study in bold, and underneath any given monoid M, we list maps ω for which we ran simulations (each pair (M, ω) was simulated for $n = 5, 10, 15, ..., 40$ in order to get a sense for the limiting distribution as $n \to \infty$). Based on this data for increasing values of n, our conjectured limiting distributions for pairs (M, ω) are listed in the right-most column of each table. Additionally, when writing down the mapping of elements for any ω, we write $a_1 : a_2$ to represent that $\omega(a_1) = a_2$. In each entry, we list the probability that a closed diagram D (chosen uniformly at random) evaluates to each element in M (these probabilities are listed in the format "element : probability").

	n = 5	n = 10	n = 15	n = 20	n = 25	n = 30	n = 35	n = 40	conjectured limit as n --> infinity
1) M = {1, a, a^2 : a^2 = a^3}									
w_0: {1:a^2; a^2:a; a:a}	{a^2: 0.650872, a: 0.349128}	{a^2: 0.750949, a: 0.249051}	{a^2: 0.849231, a: 0.150769}	{a^2: 0.907836, a: 0.092164}	{a^2: 0.944652, a: 0.055348}	{a^2: 0.968932, a: 0.031068}	{a^2: 0.982921, a: 0.017079}	{a^2: 0.988948, a: 0.011052}	pr(a^2) --> 1
w_{00}: {1:a^2; a: a; a^2: a}	{a^2: 0.662648, a: 0.337352}	{a^2: 0.751533, a: 0.248467}	{a^2: 0.850941, a: 0.149059}	{a^2: 0.903081, a: 0.096919}	{a^2: 0.945184, a: 0.054816}	{a^2: 0.967029, a: 0.032971}	{a^2: 0.981323, a: 0.018677}	{a^2: 0.98994, a: 0.01006}	pr(a^2) --> 1
w_1: {1:a; a:a; a^2: 1}	{a^2: 0.476922, a: 0.444063, 1: 0.079015}	{a^2: 0.631127, a: 0.289408, 1: 0.079465}	{a^2: 0.743448, a: 0.205158, 1: 0.051394}	{a^2: 0.818579, a: 0.147359, 1: 0.034062}	{a^2: 0.890126, a: 0.091823, 1: 0.018051}	{a^2: 0.924092, a: 0.064179, 1: 0.011729}	{a^2: 0.950026, a: 0.042756, 1: 0.007218}	{a^2: 0.968773, a: 0.027099, 1: 0.004128}	pr(a^2) --> 1
w_2: {1:a^2; a^2: a: a: 1}	{a^2: 0.666135, a: 0.244181, 1: 0.089684}	{a^2: 0.76548, a: 0.151128, 1: 0.083392}	{a^2: 0.822815, a: 0.108064, 1: 0.069121}	{a^2: 0.897694, a: 0.061132, 1: 0.041174}	{a^2: 0.935748, a: 0.038987, 1: 0.025265}	{a^2: 0.958819, a: 0.025127, 1: 0.016054}	{a^2: 0.972773, a: 0.016907, 1: 0.01032}	{a^2: 0.982167, a: 0.01147, 1: 0.006363}	pr(a^2) --> 1
w_a: {1:a^2; a:a; a^2:1}	{a^2: 0.725896, a: 0.065635, 1: 0.208469}	{a^2: 0.773515, a: 0.093398, 1: 0.133087}	{1: 0.079468, a: 0.070021, a^2: 0.850511}	{a^2: 0.903715, a: 0.048339, 1: 0.047946}	{a^2: 0.938704, a: 0.032023, 1: 0.029273}	{a^2: 0.959587, a: 0.021356, 1: 0.019057}	{a^2: 0.97472, a: 0.013748, 1: 0.011532}	{a^2: 0.98376, a: 0.008861, 1: 0.007379}	pr(a^2) --> 1
w_4: {1:a; a:1; a^2:a^2}	{a^2: 0.445385, a: 0.338102, 1: 0.216513}	{a^2: 0.638158, a: 0.234797, 1: 0.127045}	{a^2: 0.798854, a: 0.142609, 1: 0.058537}	{a^2: 0.889424, a: 0.082183, 1: 0.028393}	{a^2: 0.936768, a: 0.048938, 1: 0.014294}	{a^2: 0.968268, a: 0.025108, 1: 0.006624}	{a^2: 0.982714, a: 0.014279, 1: 0.003007}	{a^2: 0.99099, a: 0.007593, 1: 0.001417}	pr(a^2) --> 1
w_5: {1:a^2; a:1; a^2:a^2}	{a^2: 1.0}	{a^2: 1.0}							
w_6: {1:a; a:a^2; a^2:1}	{a^2: 0.658789, a: 0.229278, 1: 0.111933}	{a^2: 0.72114, a: 0.185704, 1: 0.093156}	{a^2: 0.80091, a: 0.14116, 1: 0.05793}	{a^2: 0.864567, a: 0.100428, 1: 0.035005}	{a^2: 0.904982, a: 0.072885, 1: 0.022133}	{a^2: 0.938831, a: 0.048161, 1: 0.013008}	{a^2: 0.960022, a: 0.032097, 1: 0.007881}	{a^2: 0.974801, a: 0.020796, 1: 0.004403}	pr(a^2) --> 1
w_7: {1:a; a:1; a^2:1}	{a^2: 0.331978, a: 0.395733, 1: 0.272289}	{ a^2: 0.444316, a: 0.323959, 1: 0.231725}	{a^2: 0.546153, a: 0.293213, 1: 0.160634}	{a^2: 0.641097, a: 0.243923, 1: 0.11498}	{1: 0.071865, a: 0.186508, a^2: 0.741627}	{1: 0.050391, a: 0.145335, a^2: 0.804274}	{1: 0.033654, a: 0.111738, a^2: 0.854608}	{a^2: 0.895958, a: 0.082452, 1: 0.02159}	pr(a^2) --> 1
w_8: {1: a; a:1; a^2:a}	{a^2: 0.334122, a: 0.437194, 1: 0.228684}	{a^2: 0.49072, a: 0.362027, 1: 0.147253}	{a^2: 0.655025, a: 0.263448, 1: 0.081527}	{a^2: 0.75313, a: 0.197023, 1: 0.049847}	{a^2: 0.841093, a: 0.131347, 1: 0.02756}	{a^2: 0.897287, a: 0.087071, 1: 0.015642}	{a^2: 0.938441, a: 0.053219, 1: 0.00834}	{a^2: 0.955588, a: 0.038517, 1: 0.005895}	pr(a^2) --> 1
w_a^2: {1:a^2; a:1; a^2: 1}	{a^2: 0.7182, 1: 0.2818}	{a^2: 0.774605, 1: 0.225395}	{a^2: 0.824733, 1: 0.175267}	{1: 0.113176, a^2: 0.886824}	{a^2: 0.921835, 1: 0.078165}	{1: 0.05384, a^2: 0.94616}	{1: 0.03124, a^2: 0.96876}	{a^2: 0.977707, 1: 0.022293}	pr(a^2) --> 1
w_10: {1:a^2; a:a^2; a^2:1}	{a^2: 0.721054, 1: 0.278946}	{a^2: 0.761897, 1: 0.238103}	{1: 0.170883, a^2: 0.829117}	{a^2: 0.892256, 1: 0.107744}	{a^2: 0.926933, 1: 0.073067}	{a^2: 0.951046, 1: 0.048954}	{a^2: 0.965462, 1: 0.034538}	{a^2: 0.976798, 1: 0.023202}	pr(a^2) --> 1
2) Consider commutative monoids for |M| = 2									
(a) M = {1, a: a^2 = 1}	{1.0: 0.499422, a: 0.500578}	{1: 0.500002, a: 0.499998}	{1.0: 0.499032, a: 0.500968}	{1.0: 0.500722, a: 0.499278}	{1.0: 0.498848, a: 0.501152}	{1.0: 0.500162, a: 0.499838}	{1.0: 0.499872, a: 0.500128}	{1.0: 0.500072, a: 0.499928}	pr(1) --> 1/2, pr (a) --> 1/2
(b) M = {1, a: a^2 = a}	{1: 0.284163, a: 0.715837}	{1: 0.202396, a: 0.797604}	{1: 0.163118, a: 0.836882}	{1: 0.113571, a: 0.886429}	{1: 0.072654, a: 0.927346}	{1: 0.052292, a: 0.947708}	{1: 0.03288, a: 0.96712}	{1: 0.021598, a: 0.978402}	pr(a) --> 1
w: {1:a; a:1}									
3) M = {1, a, a^2 : a^3 = 1}									
w_1: {1:a; a: a^2; a^2:a}	{a^2: 0.431393, a: 0.353779, 1.0: 0.214828}	{1.0: 0.285827, a: 0.344178, a^2: 0.369995}	{a^2: 0.353863, a: 0.332482, 1.0: 0.313655}	{a^2: 0.343536, a: 0.329563, 1.0: 0.326901}	{1.0: 0.330975, a: 0.330926, a^2: 0.338099}	{a^2: 0.334714, a: 0.332898, 1.0: 0.332388}	{1.0: 0.333107, a: 0.332511, a^2: 0.334382}	{1.0: 0.332647, a: 0.333754, a^2: 0.333599}	uniform
w_2: {1:a; a:a; a^2:a^2}	{a^2: 0.44564, a: 0.415279, 1.0: 0.139081}	{1.0: 0.289825, a: 0.311024, a^2: 0.399151}	{1.0: 0.324397, a: 0.321244, a^2: 0.354359}	{1.0: 0.332653, a: 0.330235, a^2: 0.337112}	{a^2: 0.336321, a: 0.329282, 1.0: 0.334397}	{a^2: 0.335824, a: 0.330019, 1.0: 0.334157}	{a^2: 0.331982, a: 0.332937, 1.0: 0.335081}	{a^2: 0.333373, a: 0.333695, 1.0: 0.332932}	uniform
w_a: {1:a; a:a^2; a^2:a^2}	{a^2: 0.511238, a: 0.235684, 1.0: 0.253078}	{1.0: 0.326284, a: 0.286002, a^2: 0.387714}	{a^2: 0.356899, a: 0.312582, 1.0: 0.330519}	{a^2: 0.343189, a: 0.322486, 1.0: 0.334325}	{1.0: 0.334177, a: 0.327832, a^2: 0.337991}	{1.0: 0.333641, a: 0.332324, a^2: 0.334035}	{1.0: 0.33418, a: 0.331886, a^2: 0.333934}	{1.0: 0.333256, a: 0.333527, a^2: 0.333217}	uniform
w_4: {1:a; a:a^2; a^2:1}	{a^2: 0.389908, a: 0.284914, 1: 0.325178}	{a^2: 0.329842, a: 0.332216, 1.0: 0.337942}	{1: 0.33311, a: 0.33463, a^2: 0.33226}	{a^2: 0.334612, a: 0.331846, 1: 0.333542}	{1.0: 0.333724, a: 0.33332, a^2: 0.332956}	{1.0: 0.332602, a: 0.332982, a^2: 0.334416}	{a^2: 0.334474, a: 0.332568, 1.0: 0.332958}	{1.0: 0.332348, a: 0.33419, a^2: 0.333462}	uniform
w_5: {1:a; a:a; a^2: 1}	{1.0: 0.195454, a: 0.449244, a^2: 0.355302}	{1: 0.286832, a: 0.377808, a^2: 0.33536}	{a^2: 0.343818, a: 0.348274, 1.0: 0.307908}	{1.0: 0.320219, a: 0.33742, a^2: 0.342361}	{a^2: 0.338959, a: 0.332356, 1.0: 0.328685}	{a^2: 0.335304, a: 0.332034, 1.0: 0.332662}	{1.0: 0.333953, a: 0.332074, a^2: 0.333973}	{a^2: 0.333579, a: 0.332661, 1.0: 0.33376}	uniform
w_6: {1:a; a:a; a^2:a}	{a^2: 0.387557, a: 0.498449, 1.0: 0.113994}	{1.0: 0.240028, a: 0.369552, a^2: 0.39042}	{a^2: 0.372086, a: 0.332197, 1.0: 0.295717}	{a^2: 0.353132, a: 0.323713, 1.0: 0.323155}	{a^2: 0.339416, a: 0.328455, 1.0: 0.332129}	{1.0: 0.333967, a: 0.329311, a^2: 0.336722}	{a^2: 0.334092, a: 0.331841, 1.0: 0.334067}	{a^2: 0.333793, a: 0.332568, 1.0: 0.333639}	uniform
w_7: {1:a; a:1; a^2:1}	{1: 0.35497, a: 0.399488, a^2: 0.245542}	{a^2: 0.273618, a: 0.388838, 1.0: 0.337544}	{a^2: 0.302938, a: 0.370754, 1.0: 0.326308}	{1: 0.321254, a: 0.357877, a^2: 0.320869}	{1.0: 0.326101, a: 0.34158, a^2: 0.332319}	{a^2: 0.334142, a: 0.337789, 1.0: 0.328069}	{1: 0.331287, a: 0.33456, a^2: 0.334153}	{1.0: 0.330865, a: 0.33446, a^2: 0.334675}	uniform
w_8: {1:a; a:1; a^2:a}	{1: 0.304515, a: 0.439552, a^2: 0.255933}	{1: 0.295633, a: 0.405092, a^2: 0.299275}	{a^2: 0.332643, a: 0.364765, 1.0: 0.302592}	{1.0: 0.315124, a: 0.344804, a^2: 0.340072}	{a^2: 0.340907, a: 0.333897, 1.0: 0.325196}	{a^2: 0.33682, a: 0.332746, 1.0: 0.330434}	{1.0: 0.332375, a: 0.33238, a^2: 0.335245}	{a^2: 0.335117, a: 0.332007, 1.0: 0.332876}	uniform
4) M = {1, a, a^2 : a^3 = a}									
w_1: {1:a, a:1, a^2:a}	{a^2: 0.258215, a: 0.50045, 1: 0.241335}	{a^2: 0.347267, a: 0.498848, 1: 0.153885}	{a^2: 0.390634, a: 0.500032, 1: 0.109334}	{a^2: 0.425894, a: 0.499758, 1: 0.074348}	{a^2: 0.453539, a: 0.50005, 1: 0.046411}	{1: 0.029942, a: 0.499902, a^2: 0.470156}	{a^2: 0.481861, a: 0.50005, 1: 0.018089}	{1: 0.012573, a: 0.499902, a^2: 0.487525}	pr(1) --> 0, pr(a), pr(a^2) --> 1/2
w_2: {1:a, a:a^2, a^2:1}	{a^2: 0.453932, a: 0.425377, 1: 0.120691}	{a^2: 0.431847, a: 0.457586, 1: 0.110567}	{a^2: 0.448042, a: 0.477921, 1: 0.074037}	{a^2: 0.464466, a: 0.4895, 1: 0.046034}	{a^2: 0.477845, a: 0.495414, 1: 0.026741}	{a^2: 0.485585, a: 0.497328, 1: 0.017087}	{a^2: 0.48987, a: 0.49954, 1: 0.01059}	{a^2: 0.494244, a: 0.499448, 1: 0.006308}	pr(1) --> 0, pr(a), pr(a^2) --> 1/2
5) M = {1, a, a^2, a^3 : a^4 = a}									
w_1: {1:a; a:a; a^2:1; a^a:a^2}	{a^3: 0.107868, a^2: 0.365589, a: 0.4412, 1: 0.085343}	{a^3: 0.223153, a^2: 0.353358, a: 0.340992, 1: 0.082497}	{a^3: 0.272787, a^2: 0.338362, a: 0.333971, 1: 0.05488}	{a^3: 0.295805, a^2: 0.335772, a: 0.333194, 1: 0.035229}	{a^3: 0.311945, a^2: 0.335355, a: 0.332007, 1: 0.020693}	{a^3: 0.320179, a^2: 0.333888, a: 0.333258, 1: 0.012675}	{a: 0.333267, a^2: 0.333618, a^3: 0.325975, 1: 0.00714}	{a: 0.332618, a^2: 0.334125, a^3: 0.328837, 1: 0.00442}	pr(1) --> 0, pr(a), pr(a^2), pr(a^3) --> 1/3
w_2: {1:a^2, a:1, a^2:a^3, a^a:a}	{a^3: 0.314233, a^2: 0.327966, a: 0.283535, 1: 0.074266}	{a: 0.291711, a^2: 0.33616, a^3: 0.302357, 1: 0.069772}	{a^3: 0.305691, a^2: 0.341801, a: 0.305832, 1: 0.046676}	{a^3: 0.310537, a^2: 0.341787, a: 0.317152, 1: 0.030524}	{a^3: 0.317879, 1: 0.018637, a: 0.324402, a^2: 0.339082}	{a^3: 0.321331, a^2: 0.336727, a: 0.330582, 1: 0.01136}	{a^3: 0.324304, a^2: 0.335562, a: 0.332826, 1: 0.007308}	{a^3: 0.327933, a^2: 0.334784, a: 0.333066, 1: 0.004217}	pr(1) --> 0, pr(a), pr(a^2), pr(a^3) --> 1/3